Optimum Design Parameters for Hypergolic Reciprocating Engines—A Mathematical Solution

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The cycle equations for hypergolic reciprocating engines are derived from basic thermodynamic principles. The calculus of variations is applied in order to obtain optimum design parameters using the criterion of a minimum specific expendables consumption. A numerical scheme is used to solve a resulting set of nonlinear equations and calculated results are presented. The results indicate that the optimum design parameters depend strongly on the pressure ratio and weakly on the ratio of specific heats of the working fluid. The optimum value of the specific expendables consumption is related to the pressure ratio in a hyperbolic fashion. Provided that the design point is in the neighborhood of the optimum point, nonoptimum designs will not suffer severely in performance.

Nomenclature

A = area; also function defined by Eq. (18a)
B = function defined by Eq. (18b)
C_f = card factor
C_L = clearance ratio = V_{cL}/V_d
C_0 = cutoff ratio = V_{c0}/V_d
C_w = flow factor
c_i = coefficients in expansion for C_0*, Eq. (26)
D = engine displacement per revolution, ft³/rev
E_m = mechanical efficiency
E_x = exhaust ratio = V_{ex}/V_d
e_i = coefficients in expansion for E_x*, Eq. (29)
G = factor = $1.98 \times 10^6\ C_w/E_mC_fRT_3$, lbm/hp hr
hp = power output, horsepower
k = ratio of specific heat at constant pressure to that at constant volume
M = molecular mass, lbm/lb mol
M_{adm} = mass admitted per cycle, lbm/cycle
MEP = mean effective pressure, lbf/in.²
N = engine speed, rpm
P = pressure, lbf/in.²
P_c = combustion pressure, lbf/in.²
PR = pressure ratio = P_c/P_x
Pr = inverse pressure ratio = P_x/P_c
P_x = exhaust pressure, lbf/in.²
R = individual gas constant = $\mathcal{R}/M$
$\mathcal{R}$ = universal gas constant = 1545.81 ft·lbf/lbm·°R
SEC = specific expendables consumption, lbm/hp hr
s_i = coefficients in expansion for SEC*, Eq. (32)
T = absolute temperature, °R
V = volume
V_{cL} = clearance volume
V_{c0} = volume at inlet valve closure
V_d = displacement volume of cylinder
V_{ex} = volume at exhaust valve closure
W = mass flow rate, lb/min
α_i = functions defined by Eqs. (23a, 23b, and 23c), respectively
β_i = functions defined by Eqs. (25a, 25b, and 25c), respectively
γ_i = coefficient in expansion for C_0*, Eq. (27)
μ_i = coefficient in expansion for E_x*, Eq. (30)
ϕ_i = coefficient in expansion for SEC*, Eq. (33)

Subscripts

1,2,3,4, 5,6 = states as indicated in Fig. 1

Superscript

( )* = optimum value

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Introduction

Although reciprocating engines are widely used in heat engineering applications, their particular advantage lies in the range of small displacements and high pressures. Since these characteristics match the requirements of torpedo propulsion systems, reciprocating engines have long been employed as powerplants for torpedoes. In fact, the first self-propelled torpedoes[1] were driven by the Whitehead reciprocating engine, which operated with compressed air as the working fluid. At the present time, the most modern and sophisticated torpedoes are powered by an adaptation of the Hermann cam engine.

As in the design of any system, it is advisable to optimize as many design parameters as possible on an a priori basis. This reduces time spent in the prototype development stage.

An optimization analysis based on the thermodynamic cycle of reciprocating engines is presented. This permits the calculation of values for two design parameters yielding a minimum specific expendables consumption, for given values of the other cycle parameters.

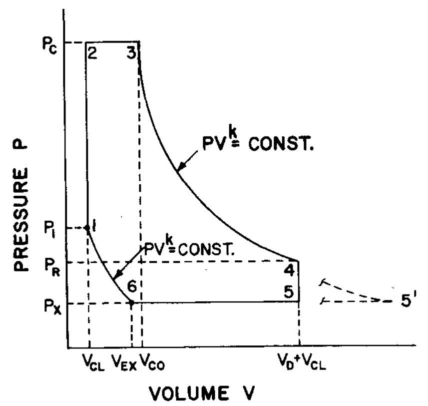


Fig. 1 Pressure-volume diagram for a reciprocating engine

Thermodynamic Cycle

Quite generally, the idealized thermodynamic, pressure-volume diagram is shown in Fig. 1. The cycle given applies to both internal and external combustion powerplants. Figure 1 is sometimes called an indicator diagram.[2]

In the case of an internal combustion engine, admission of the oxidant and fuel occurs approximately from state 1 to state 2. Constant pressure is maintained during combustion from state 2 to state 3. For external combustion, admission of the working fluid takes place fully between states 1 and 3.

For purposes of analysis, a reversible, polytropic expansion process is assumed between states 3 and 4. It is further assumed that the polytropic index n is equal to the ratio of specific heats k, thereby rendering the expansion process isentropic. The exhaust valve opens at state 4 resulting in incomplete expansion. This is done at a point where there is sufficient pressure left to overcome friction drag in the real engine. Also, this permits a smaller stroke for the piston and a more compact size for the engine. The working fluid is exhausted between states 4 and 6 at which point the exhaust valve closes and recompression occurs from state 6 to state 1. Under certain conditions it is desirable to design the engine such that states 1 and 2 coincide.

Derivation of Equations

The cycle equations are fully derived here for the sake of completeness and ease of reference. By definition,[2] the idealized mean effective pressure (MEP) is equal to the area enclosed by the curve 123456 in the pressure-volume diagram, Fig. 1, divided by the displacement volume V_d. The area 123456 may be evaluated as follows:

$$A_{123456} = A_{23} + A_{34} - A_{65} - A_{16} \tag{1}$$

where A_{23} represents the area beneath the line 23, etc. It follows on inspection that

$$A_{23} = P_c(V_{c0} - V_{cL}) \tag{2}$$

and

$$A_{65} = P_x(V_d + V_{cL} - V_{ex}) \tag{3}$$

The remaining two areas may be found by evaluating the integral of PdV over the appropriate limits of integration, using the fact that PV^k is a constant along the processes 34 and 61. That is,

$$A_{34} = \int_{V_{c0}}^{V_d+V_{cL}} PdV = \int_{V_{c0}}^{V_d+V_{cL}} P_cV_{c0}^k \frac{dV}{V^k}$$

since $PV^k = \text{const} = P_cV_{c0}^k$. Thus,

$$A_{34} = P_cV_{c0}^k[(V_d + V_{cL})^{1-k} - V_{c0}^{1-k}]/(1 - k) \tag{4}$$

Likewise,

$$A_{16} = \int_{V_{cL}}^{V_{ex}} PdV = \int_{V_{cL}}^{V_{ex}} P_xV_{ex}^k \frac{dV}{V^k}$$

since $PV^k = \text{const} = P_xV_{ex}^k$. Thus,

$$A_{16} = P_xV_{ex}^k[V_{ex}^{1-k} - V_{cL}^{1-k}]/(1 - k) \tag{5}$$

Substitution of the expressions given in Eqs. (2–5) into Eq. (1) and some algebraic manipulation yield the following relation for the area of the pressure-volume diagram:

$$A_{123456} = P_c\left\{V_{c0} - V_{cL} + \left(\frac{V_{c0}}{k-1}\right) \times \left[1 - \left(\frac{V_{c0}}{V_d + V_{cL}}\right)^{k-1}\right]\right\} + P_x\left\{V_{ex} - V_{cL} - V_d + \left(\frac{V_{ex}}{k-1}\right)\left[1 - \left(\frac{V_{ex}}{V_{cL}}\right)^{k-1}\right]\right\} \tag{6}$$

Now, using the definition of the MEP, i.e.,

$$MEP = A_{123456}/V_d \tag{7}$$

together with the following definitions of nondimensional ratios

$$C_0 = V_{c0}/V_d \tag{8a}$$

$$C_L = V_{cL}/V_d \tag{8b}$$

$$E_x = V_{ex}/V_d \tag{8c}$$

it can easily be seen that

$$MEP = P_c\left\{C_0 - C_L + \left(\frac{C_0}{k-1}\right)\left[\left(1 - \frac{C_0}{C_L+1}\right)^{k-1}\right]\right\} + P_x\left\{E_x - C_L - 1 + \left(\frac{E_x}{k-1}\right)\left[\left(1 - \frac{E_x}{C_L}\right)^{k-1}\right]\right\} \tag{9}$$

The power output of the engine is given by the standard formula,

$$hp = 144E_mC_f(MEP)ND/33000 \tag{10}$$

where the numerical factors have been included to force the units of the terms (see Nomenclature).

Since the optimization analysis is to be made on a minimum specific expendables basis, it is necessary to know the mass flow rate per cycle. This is found by first subtracting the residual mass at state 1 from the total mass at state 3 (Fig. 1). Thus,

$$M_{\text{adm}} = 144C_w[P_cV_{c0}/RT_3 - P_1V_{cL}/RT_1] \tag{11}$$

Since the mass of working fluid is constant from state 6 to state 1, it follows that

$$P_1V_{cL}/T_1 = P_xV_{ex}/T_6 \tag{12}$$

Further, since an open system prevails from state 6 back to state 5 (or 5′), it is possible to write

$$T_6 = T_{5'} \tag{13}$$

Combining Eqs. (11–13) yields

$$M_{\text{adm}} = C_w[P_cV_{c0}/T_3 - P_xV_{ex}/T_{5'}]/R \tag{14}$$

Finally, since the expansion from state 3 to state 4 (or 5′) is assumed to be isentropic, it follows that

$$T_{5'} = T_3[P_x/P_c]^{(k-1)/k} \tag{15}$$

Elimination of $T_{5'}$ between Eqs. (14) and (15) together with some algebraic manipulation yield the equation for the mass flow per cycle:

$$M_{\text{adm}} = 144C_wP_c[V_{c0} - V_{ex}(P_x/P_c)^{1/k}]/RT_3 \tag{16}$$

To obtain the mass flow rate, it is necessary to multiply M_{adm} by engine speed and to recall Eqs. (8a) and (8c). Hence, the expression for the mass flow rate becomes

$$W = 144C_wNDP_c[C_0 - E_x(P_x/P_c)^{1/k}]/RT_3 \tag{17}$$

It is a simple matter now to form a relation for specific expendables consumption (SEC) which is defined as the mass flow rate per power output. Equations (9, 10, and 17) may be combined to produce the following working equations:

$$SEC = \frac{1.98 \times 10^6C_w[C_0 - E_x(P_x/P_c)^{1/k}]}{E_mC_fRT_3[A + BP_x/P_c]} \tag{18}$$

where

$$A = C_0 - C_L + C_0\{1 - [C_0/(C_L + 1)]^{k-1}\}/(k - 1) \tag{18a}$$

and

$$B = E_x - C_L - 1 + E_x[1 - (E_x/C_L)^{k-1}]/(k - 1) \tag{18b}$$

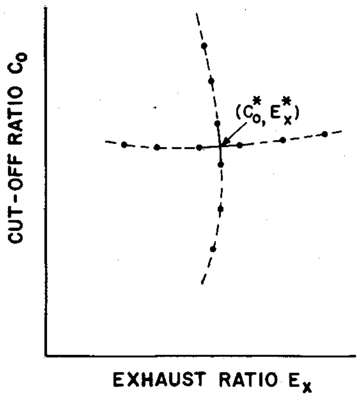


Fig. 2 Linear interpolation for C_o* and E_x* (not to scale).

Statement of the Problem

The problem is to find that set of engine design parameters, C_0, C_L, and E_x, which will yield a minimum value for SEC subject to given values of pressure ratio P_c/P_x, combustion temperature T_3, specific heat ratio k, and gas constant R, for the working fluid.

From Eqs. (18, 18a, and 18b), it may be seen that the functions A and B depend primarily upon the engine design parameters: cutoff ratio C_0; clearance ratio C_L; and exhaust ratio E_x. The specific heat ratio k of the working fluid also appears. It should be observed that the displacement of the engine does not influence the specific expendables consumption.

A significant simplification of the problem occurs when it is realized that the clearance ratio C_L ought to be made as small as possible, with the ideal case occurring when $C_L = 0$. Hence, the clearance volume is not a proper variable to be determined through optimization. Thus, it can be taken as a constant along with the pressure ratio, the combustion temperature, etc.

In addition, physical reasoning dictates that the allowable values for cutoff ratio C_0 and exhaust ratio E_x must lie in the open interval between zero and one, i.e., $0.0 < C_0, E_x < 1.0$.

Solution

Optimization Analysis

The calculus of variations[3] is used to effect the optimization analysis. For the purposes of this section, it is convenient to rewrite Eq. (18) in the form

$$SEC = G(C_0 - E_x Pr^{1-k})/(A + BPr) \tag{19}$$

Partial derivatives of SEC must be taken with respect to both C_0 and E_x, and the resulting expressions equated to zero. That is,

$$\left.\frac{\partial(SEC)}{\partial(C_0)}\right|_{G,Pr\ldots} = 0 \tag{20}$$

and

$$\left.\frac{\partial(SEC)}{\partial(E_x)}\right|_{G,Pr\ldots} = 0 \tag{21}$$

The optimum values for C_0 and E_x may be found from the simultaneous solution to Eqs. (20) and (21).

Performing the indicated differentiation in Eq. (20) results in the equation

$$A + BPr = (C_0 - E_x Pr^{1/k}) \times \left[1 + \frac{1}{k-1} - \frac{kC_0^{k-1}}{(k-1)(C_L+1)^{k-1}}\right] \tag{22}$$

After multiplying out the terms and recombining them, the first of the final optimization equations is found;

$$\alpha_1 C_0^k + \alpha_2 C_0^{k-1} + \alpha_3 = 0 \tag{23}$$

where

$$\alpha_1 = -1/(C_L + 1)^{k-1} \tag{23a}$$

$$\alpha_2 = kE_x Pr^{1/k}/[(k-1)(C_L+1)^{k-1}] \tag{23b}$$

$$\alpha_3 = C_L - BPr - kE_x Pr^{1/k}/(k-1) \tag{23c}$$

with B having been given in Eq. (18b).

The following expression results from the differentiation of Eq. (21):

$$-(A + BPr)Pr^{1/k} = Pr(C_0 - E_x Pr^{1/k}) \times \left[1 + \frac{1}{k-1} - \frac{kE_x^{k-1}}{(k-1)C_L^{k-1}}\right] \tag{24}$$

The second optimization equation is found as before:

$$\beta_1 E_x^k + \beta_2 E_x^{k-1} + \beta_3 = 0 \tag{25}$$

where

$$\beta_1 = -Pr^{(k+1)/k}/C_L^{k-1} \tag{25a}$$

$$\beta_2 = kC_0 Pr/[(k-1)C_L^{k-1}] \tag{25b}$$

and

$$\beta_3 = -APr^{1/k} + (C_L + 1)Pr^{(k+1)/k} - kC_0 Pr/(k-1) \tag{25c}$$

with A having been given in Eq. (18a).

The nonlinearity of Eqs. (23) and (25) is apparent, since α_2 and α_3 both depend on E_x, while β_2 and β_3 both depend on C_0. Solution of these equations is further complicated by the fact that the exponents containing k are noninteger. These characteristics make it impossible, in practice, to obtain an exact solution in closed mathematical form, and necessitate the use of an appropriate numerical method.

Table 1 Optimum values of cutoff ratio

	Pressure ratio, PR							
c_p/c_v	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
a) for clearance ratio of 0.05								
1.20	0.592	0.491	0.421	0.370	0.331	0.300	0.275	0.254
1.21	0.594	0.494	0.424	0.373	0.334	0.303	0.278	0.257
1.22	0.596	0.497	0.427	0.377	0.337	0.306	0.281	0.260
1.23	0.599	0.499	0.430	0.380	0.340	0.309	0.284	0.263
1.24	0.601	0.502	0.434	0.383	0.344	0.312	0.287	0.266
1.25	0.604	0.505	0.437	0.386	0.347	0.315	0.290	0.269
1.26	0.606	0.508	0.440	0.389	0.350	0.319	0.293	0.272
1.27	0.609	0.511	0.443	0.392	0.353	0.322	0.296	0.275
1.28	0.611	0.514	0.446	0.395	0.356	0.325	0.299	0.278
1.29	0.613	0.516	0.449	0.398	0.359	0.328	0.302	0.281
1.30	0.616	0.519	0.452	0.401	0.362	0.331	0.305	0.284
b) for clearance ratio of 0.10								
1.20	0.620	0.513	0.442	0.389	0.348	0.315	0.288	0.266
1.21	0.623	0.518	0.445	0.392	0.351	0.318	0.291	0.269
1.22	0.625	0.520	0.448	0.395	0.354	0.321	0.294	0.272
1.23	0.627	0.523	0.451	0.398	0.357	0.324	0.298	0.275
1.24	0.630	0.526	0.454	0.401	0.360	0.327	0.301	0.278
1.25	0.632	0.529	0.457	0.404	0.363	0.330	0.304	0.282
1.26	0.635	0.532	0.460	0.407	0.366	0.334	0.307	0.285
1.27	0.637	0.535	0.463	0.410	0.369	0.337	0.310	0.288
1.28	0.640	0.537	0.466	0.413	0.372	0.340	0.313	0.291
1.29	0.642	0.540	0.469	0.417	0.376	0.343	0.316	0.294
1.30	0.644	0.543	0.472	0.420	0.379	0.346	0.319	0.297
c) for clearance ratio of 0.20								
1.20	0.677	0.562	0.483	0.425	0.380	0.345	0.316	0.292
1.21	0.680	0.565	0.486	0.428	0.383	0.348	0.319	0.295
1.22	0.682	0.568	0.489	0.431	0.387	0.351	0.322	0.298
1.23	0.685	0.571	0.492	0.434	0.390	0.354	0.325	0.301
1.24	0.687	0.574	0.495	0.438	0.393	0.357	0.328	0.304
1.25	0.689	0.577	0.499	0.441	0.396	0.360	0.331	0.307
1.26	0.692	0.580	0.502	0.444	0.399	0.363	0.334	0.310
1.27	0.694	0.582	0.505	0.447	0.402	0.367	0.337	0.313
1.28	0.697	0.585	0.508	0.450	0.405	0.370	0.341	0.316
1.29	0.699	0.588	0.511	0.453	0.408	0.373	0.344	0.319
1.30	0.701	0.591	0.514	0.456	0.412	0.376	0.347	0.322

Numerical Method

The general approach may be summarized as follows. All allowable pairs of values for C_0 and E_x which satisfy Eq. (23) and Eq. (25), separately, are found using the method of false position,[4] or the equivalent secant method.[5] These solutions trace two curves in the (C_0,E_x) plane. The intersection of the two curves defines the pair of values for C_0 and E_x which satisfy Eqs. (23) and (25), simultaneously.

Instead of smooth curves, the loci of these curves, that is, many discrete, closely-spaced points, are obtained. This is done, in the case of Eq. (23), by selecting many values for E_x and solving the resulting expressions for the corresponding values of C_0. The same is done with Eq. (25) except that the roles of C_0 and E_x are reversed. Linear interpolation is then used with the four points closest to the intersection, as illustrated in Fig. 2. Since the curves are relatively flat in the region of intersection and the points are closely spaced, this method yields very accurate results. The entire mathematical operation has been programmed onto the CDC-3200 electronic digital computer in the 3200-FORTRAN language, and the results are presented in Tables 1–3.

Discussion of Results

General Discussion

It may be seen from an examination of Eqs. (18a, 18b, 23, and 25) that the optimum pair of values for the cutoff ratio and the exhaust ratio depends only on the pressure ratio, the specific heat ratio of the working fluid, and the clearance volume. The last of these is normally made as small as possible within practical limits to produce the most compact engine. The molecular weight and temperature of the working fluid, together with the empirical factors C_w, E_m, and C_x, enter the calculations only when values for the specific expendables consumption are required.

Tables 1 and 2 contain the optimum values of the design parameters, the cutoff ratio and the exhaust ratio, for values of the specific heat ratio from 1.20–1.30 in increments of 0.01 and for values of the pressure ratio from 2.0–5.5 in increments of 0.5, each given for values of the clearance ratio of 0.05, 0.10, and 0.20. Table 3 shows the corresponding optimum values of the specific expendables consumption over the same range of specific heat ratio and pressure ratio, but for a value of the consumption factor of 10.0.

The range of values chosen for the specific heat ratio, 1.20–1.30, covers the values available from present fuel-oxidizer combinations. Selected values of the specific heat ratio[6] are given in Table 4. Even in those applications where it is necessary to add water as a diluent, the resulting effective specific heat ratio will lie within the quoted range.

The results of the analysis show that the optimum cutoff and exhaust ratios, i.e., those values which yield a minimum specific expendables consumption, depend strongly on the pressure ratio, with the dependency of the optimum cutoff ratio being the greatest at the lower values of the pressure ratios corresponding to deep-depth operation. Both the optimum cutoff and exhaust ratio turn out to be linearly related to the clearance ratio and specific heat ratio. These relations can be observed graphically in Figs. 3a and 3b.

The optimum specific expendables consumption rises rapidly as the pressure ratio decreases and is weakly and linearly dependent upon the specific heat ratio whereas the consumption factor has a direct linear effect upon the optimum consumption. Figure 4 illustrates these relationships.

Table 2 Optimum values of exhaust ratio

	Pressure ratio, *PR*							
c_p/c_v	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
a) for clearance ratio of 0.05								
1.20	0.089	0.108	0.125	0.143	0.159	0.176	0.192	0.208
1.21	0.089	0.107	0.124	0.141	0.158	0.174	0.190	0.206
1.22	0.088	0.106	0.124	0.140	0.157	0.173	0.188	0.204
1.23	0.088	0.106	0.123	0.139	0.155	0.171	0.187	0.202
1.24	0.088	0.105	0.122	0.138	0.154	0.170	0.185	0.200
1.25	0.087	0.104	0.121	0.137	0.153	0.168	0.183	0.198
1.26	0.087	0.104	0.120	0.136	0.151	0.166	0.181	0.196
1.27	0.086	0.103	0.119	0.135	0.150	0.165	0.179	0.194
1.28	0.086	0.103	0.118	0.134	0.149	0.163	0.177	0.191
1.29	0.087	0.102	0.118	0.133	0.147	0.162	0.176	0.189
1.30	0.085	0.101	0.117	0.132	0.146	0.160	0.174	0.187
b) for clearance ratio of 0.10								
1.20	0.179	0.215	0.251	0.285	0.319	0.352	0.384	0.416
1.21	0.178	0.214	0.249	0.283	0.316	0.349	0.381	0.412
1.22	0.177	0.213	0.247	0.281	0.313	0.345	0.377	0.408
1.23	0.176	0.211	0.245	0.279	0.311	0.342	0.373	0.404
1.24	0.175	0.210	0.244	0.276	0.308	0.339	0.370	0.400
1.25	0.174	0.209	0.242	0.274	0.305	0.336	0.366	0.395
1.26	0.174	0.208	0.240	0.272	0.303	0.333	0.362	0.391
1.27	0.173	0.206	0.239	0.270	0.300	0.380	0.359	0.387
1.28	0.172	0.205	0.237	0.268	0.298	0.327	0.355	0.383
1.29	0.171	0.204	0.235	0.266	0.295	0.323	0.351	0.379
1.30	0.170	0.203	0.234	0.263	0.292	0.320	0.348	0.375
c) for clearance ratio of 0.20								
1.20	0.357	0.430	0.501	0.570	0.638	0.703	0.768	0.832
1.21	0.355	0.428	0.498	0.566	0.632	0.697	0.761	0.824
1.22	0.354	0.425	0.494	0.561	0.627	0.691	0.754	0.816
1.23	0.352	0.423	0.491	0.557	0.622	0.685	0.746	0.807
1.24	0.351	0.420	0.488	0.553	0.616	0.678	0.739	0.799
1.25	0.349	0.418	0.484	0.548	0.611	0.672	0.732	0.791
1.26	0.347	0.415	0.481	0.544	0.606	0.666	0.725	0.783
1.27	0.346	0.413	0.477	0.540	0.600	0.659	0.717	0.774
1.28	0.344	0.410	0.474	0.535	0.595	0.653	0.710	0.766
1.29	0.343	0.408	0.471	0.531	0.590	0.647	0.703	0.758
1.30	0.341	0.405	0.467	0.527	0.584	0.640	0.695	0.749

Table 3 Optimum values of specific expendables consumption for consumption factor of 10.0

	Pressure ratio, *PR*							
c_p/c_v	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
1.20	15.281	11.776	9.960	8.846	8.084	7.523	7.088	6.738
1.21	15.314	11.810	9.994	8.881	8.119	7.558	7.123	6.774
1.22	15.348	11.844	10.029	8.915	8.154	7.593	7.159	6.809
1.23	15.381	11.878	10.063	8.950	8.189	7.628	7.194	6.845
1.24	15.415	11.912	10.097	8.985	8.224	7.664	7.230	6.881
1.25	15.448	11.946	10.132	9.020	8.259	7.699	7.265	6.916
1.26	15.482	11.980	10.166	9.054	8.294	7.734	7.300	6.952
1.27	15.515	12.014	10.201	9.089	8.329	7.769	7.336	6.988
1.28	15.549	12.048	10.235	9.124	8.364	7.805	7.371	7.023
1.29	15.582	12.082	10.269	9.158	8.399	7.840	7.407	7.059
1.30	15.616	12.116	10.304	9.193	8.434	7.875	7.442	7.095

Polynomial Correlations

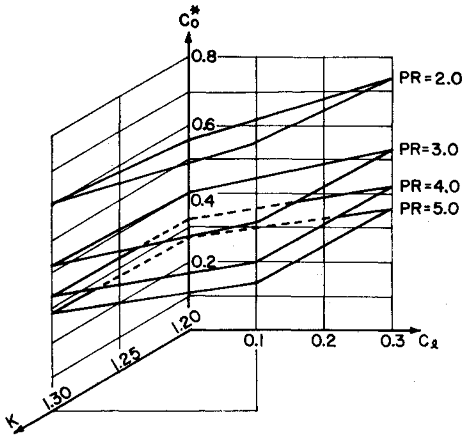
The optimum values of the cutoff and exhaust ratios and the specific expendables consumption have been correlated with the aid of the method of least squares.[7] The general forms of the polynomial expressions are presented for each function;

Optimum cutoff ratio:

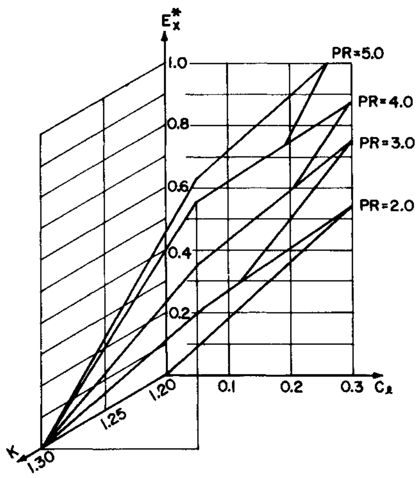
$$C_0^*(C_L,k,PR) = c_0(C_L,k) + c_1(C_L,k)\frac{1}{PR} + c_2(C_L,k)\left(\frac{1}{PR}\right)^2 + c_3(C_L,k)\left(\frac{1}{PR}\right)^3 \quad (26)$$

Table 4 Values of specific heat ratio for selected fuel-oxidizer combinations

Fuel	Oxidizer	Specific heat ratio, c_p/c_v
JP-4	99.6% H_2O_2	1.20
JP-4	Fluorine	1.22
Isopropyl alcohol	Oxygen	1.22
Hydrogen	Oxygen	1.22–1.26
JP-4	RFNA (22% NO_2)	1.23
Ammonia	Oxygen	1.23
JP-4	Oxygen	1.24
Hydrazine	90% H_2O_2	1.25
Hydrazine	Oxygen	1.25
JP-4	100% Ozone	1.25
Diborane	Fluorine	1.30



a) Cutoff



b) Exhaust

Fig. 3 Optimum ratio.

where

$$c_0(C_L,k) = \gamma_0(k) + 0.04036\ C_L \quad (27a)$$

$$c_1(C_L,k) = \gamma_1(k) + 1.280\ C_L \quad (27b)$$

$$c_2(C_L,k) = \gamma_2(k) - 0.584\ C_L \quad (27c)$$

$$c_3(C_L,k) = \gamma_3(k) + 0.294\ C_L \quad (27d)$$

with

$$\gamma_0(k) = 0.0235 + 0.195(k - 1.2) \quad (28a)$$

$$\gamma_1(k) = 1.289 + 1.030(k - 1.2) \quad (28b)$$

$$\gamma_2(k) = -0.575 - 2.62(k - 1.2) \quad (28c)$$

$$\gamma_3(k) = 0.310 + 1.50(k - 1.2) \quad (28d)$$

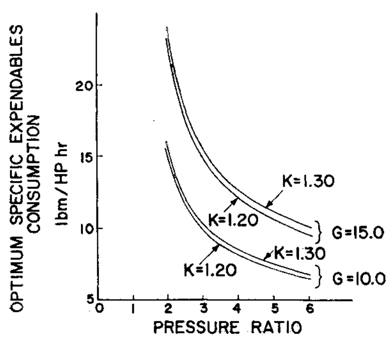


Fig. 4 Optimum specific expendables consumption.

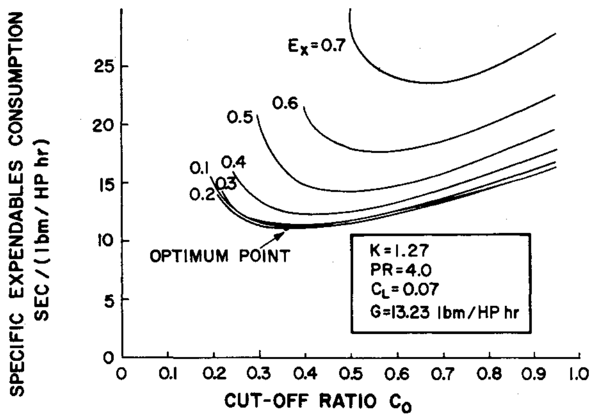


Fig. 5 Specific expendables consumption as a function of cutoff ratio with exhaust ratio as a parameter.

Optimum exhaust ratio:

$$E_x^*(C_L,k,PR) = e_0(C_L) + e_1(C_L,k)(PR - 1) + e_2(C_L,k) \times (PR - 1)^2 + e_3(C_L,k)(PR - 1)^3 \quad (29)$$

where

$$e_0(C_L) = 1.015\ C_L \quad (30a)$$

$$e_1(C_L,k) = \mu_1(k)\ C_L \quad (30b)$$

$$e_2(C_L,k) = \mu_2(k)\ C_L \quad (30c)$$

$$e_3(C_L,k) = \mu_3(k)\ C_L \quad (30d)$$

with

$$\mu_1(k) = 0.7988 - 0.7580(k - 1.2) \quad (31a)$$

$$\mu_2(k) = -0.0304 - 0.0545(k - 1.2) \quad (31b)$$

$$\mu_3(k) = 0.00184 + 0.0040(k - 1.2) \quad (31c)$$

Optimum specific expendables consumption:

$$SEC^*(G,k,PR) = s_0(G,k) + s_1(G,k)\frac{1}{PR} + s_2(G,k) \times \left(\frac{1}{PR}\right)^2 + s_3(G,k)\left(\frac{1}{PR}\right)^k \quad (32)$$

where

$$s_0(G,k) = \phi_0(k)G \quad (33a)$$

$$s_1(G,k) = \phi_1(k)G \quad (33b)$$

$$s_2(G,k) = \phi_2(k)G \quad (33c)$$

$$s_3(G,k) = \phi_3(k)G \quad (33d)$$

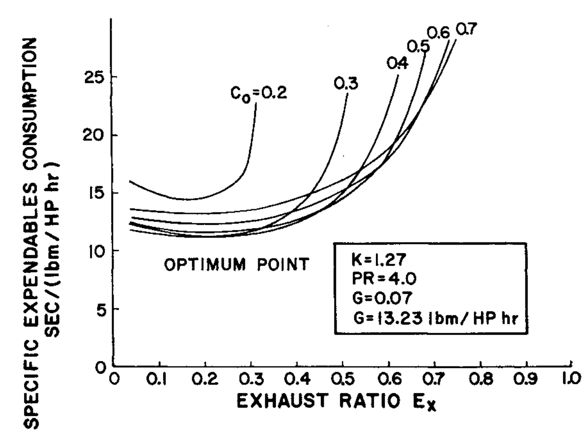


Fig. 6 Specific expendables consumption as a function of exhaust ratio with cutoff ratio as a parameter.

with

$$\phi_0(k) = 0.30585 + 0.3848(k - 1.2) \tag{34a}$$

$$\phi_1(k) = 2.3365 - 0.207(k - 1.2) \tag{34b}$$

$$\phi_2(k) = 2.8265 + 0.325(k - 1.2) \tag{34c}$$

$$\phi_3(k) = 6.0847 - 0.220(k - 1.2) \tag{34d}$$

These polynomials reproduce the exact, numerical solutions with an error or not exceeding 0.05% in all cases. It must be emphasized, however, that these correlations are strictly valid only within the following ranges:

$$1.20 \leqslant k \leqslant 1.30 \qquad 2.0 \leqslant PR \leqslant 5.5$$

$$0 \leqslant C_L \leqslant 0.25 \qquad 10 \leqslant G \leqslant 17$$

Nonoptimum Design Analysis

The nonoptimum performance characteristics are shown in Figs. 5 and 6. Figure 5 presents the specific expendables consumption in terms of cutoff ratio C_0, with exhaust ratio E_x as a parameter. Figure 6 is a cross plot of the same data, with the roles of C_0 and E_x reserved. Taking the two figures together, it may be seen that engine performance is relatively unaffected by small changes in design parameters in the neighborhood of the optimum point. However, the performance deteriorates rapidly as the departure from the optimum parameters increases, the worst cases corresponding to smaller than optimum values of the cutoff ratio and larger than optimum values of the exhaust ratio.

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